INTER-UNIVERSAL TEICHMÜLLER THEORY AS AN ANABELIAN GATEWAY TO DIOPHANTINE GEOMETRY AND ANALYTIC NUMBER THEORY (MFO-RIMS23 VERSION)

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https://www.kurims.kyoto-u.ac.jp/~motizuki/IUT%20as%20an%20
Anabelian%20Gateway%20(MFO-RIMS23%20Version).pdf

§1. Overview via a famous quote of Poincaré

(cf. [Alien]; [EssLgc], §1.5; [EssLgc], Examples 2.4.7, 2.4.8, 3.3.2; [ClsIUT], §4)

· In this talk, we give an *overview of various aspects of IUT*, many of which may be regarded as *striking examples* of the famous quote of *Poincaré* to the effect that

"*mathematics is the art of giving the same name to different things*".

— which was apparently *originally motivated* by various mathematical observations on the part of Poincaré concerning certain remarkable similarities betw'n *transformation group symmetries* of modular functions such as *theta functions*, on the one hand, and *symmetry groups* of the *hyperbolic geometry* of the *upper half-plane*, on the other — all of which are *closely related to IUT!*

· Here, we note that there are *three ways* in which this quote of *Poincaré* is related to *IUT*:

· the *original motivation* of Poincaré (mentioned above),
· the key IUT notions of *coricity/multiradiality* (cf. §2, §3),
· *new applications* of the *Galois-orbit version of IUT* (cf. §4).

· One important theme: it is possible to acquire a *survey-level* understanding of IUT using only a knowledge of such elementary topics as

· the elem. notions of *rings/fields/groups/monoids* (cf. §2),
· the elem. geom. of the *proj. line/Riemann sphere* (cf. §3).

· A more detailed exposition of IUT may be found in

· the *survey texts* [Alien], [EssLgc], as well as in
· the *videos/slides* available at the following URLs:

`https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHoriz IUT21/WS3/ExpHorizIUT21-InvitationIUT-notes.html`

`https://www.kurims.kyoto-u.ac.jp/~motizuki/ExpHoriz IUT21/WS4/ExpHorizIUT21-IUTSummit-notes.html`

§2. **Galois groups as abstract groups: the example of the N-th power map**
(cf. [EssLgc], Example 2.4.8; [EssLgc], §3.2, §3.8)

· Let R be an *integral domain* (e.g., $\mathbb{Z} \subseteq \mathbb{Q}$) equipped with the action of a *group* G, $(\mathbb{Z} \ni)$ $N \geq 2$. For simplicity, assume that $N = 1 + \cdots + 1 \neq 0 \in R$; R has *no nontrivial N-th roots of unity*. Write $R^{\triangleright} \subseteq R$ for the *multiplicative monoid* $R \setminus \{0\}$. Then let us observe that the *N-th power map* on $R^{\triangleright}$ determines an *isomorphism of multiplicative monoids* equipped with actions by G

$$G \curvearrowright R^{\triangleright} \stackrel{\sim}{\rightarrow} (R^{\triangleright})^N \ (\subseteq R^{\triangleright}) \curvearrowleft G$$

that does *not arise* from a *ring homomorphism*, i.e., it is clearly *not compatible* with *addition* (cf. our assumption on N!).

· Let ${}^{\dagger}R$, ${}^{\ddagger}R$ be *two distinct copies* of the integral domain R, equipped with respective actions by *two distinct copies* ${}^{\dagger}G$, ${}^{\ddagger}G$ of the group G. We shall use similar notation for objects with labels "†", "‡" to the previously introduced notation. Then one may use the *isomorphism of multiplicative monoids* arising from the *N-th power map* discussed above to *glue* together

$${}^{\dagger}G \curvearrowright {}^{\dagger}R \supseteq ({}^{\dagger}R^{\triangleright})^N \stackrel{\sim}{\leftarrow} {}^{\ddagger}R^{\triangleright} \subseteq {}^{\ddagger}R \curvearrowleft {}^{\ddagger}G$$

the *ring* ${}^{\dagger}R$ to the *ring* ${}^{\ddagger}R$ along the *multiplicative monoid* $({}^{\dagger}R^{\triangleright})^N \stackrel{\sim}{\leftarrow} {}^{\ddagger}R^{\triangleright}$. This gluing is *compatible* with the respective actions of ${}^{\dagger}G$, ${}^{\ddagger}G$ relative to the isomorphism ${}^{\dagger}G \stackrel{\sim}{\rightarrow} {}^{\ddagger}G$ given by forgetting the labels "†", "‡", but, since the N-th power map is *not compatible* with *addition* (!), this isomorphism ${}^{\dagger}G \stackrel{\sim}{\rightarrow} {}^{\ddagger}G$ may be regarded either as an isomorphism of (*"coric"*, i.e., *invariant* with respect to the N-th power map) *abstract groups* (cf. the notion of *"inter-universality"*, as discussed in [EssLgc], §3.2, §3.8!) or as an isomorphism of groups equipped with actions on certain *multiplicative monoids*, but *not* as an isomorphism of (*"Galois"* — cf. the *inner automorphism indeterminacies* of SGA1!) groups equipped with actions on *rings/fields*.

· The problem of *describing (certain portions of the) ring structure* of $^{\dagger}R$ in terms of the *ring structure* of $^{\ddagger}R$ — in a fashion that is *compatible* with the *gluing* and via a *single* algorithm that may be applied to the *common* (cf. *logical AND $\wedge$!*) *glued data* to reconstruct *simultaneously* (certain portions of) the ring structures of *both* $^{\dagger}R$ and $^{\ddagger}R$, up to suitable relatively mild *indeterminacies* (cf. the theory of *crystals*!) — seems (at first glance/in general) to be *hopelessly intractable* (cf. the case of $\mathbb{Z}$)!

One well-known elementary example: when $N = p$, working *modulo p* (cf. *indeterminacies*, analogy with *crystals*!), where there is a *common ring structure* that is *compatible* with the *p-th power map*!

This is precisely what is *achieved in IUT* (cf. quote of *Poincaré*!) by means of the *multiradial algorithm for the Θ-pilot* via

· *anabelian geometry* (cf. the *abstract groups* $^{\dagger}G$, $^{\ddagger}G$!);
· *the p-adic/complex logarithm, theta functions*;
· *Kummer theory*, to relate *Frob.-/étale-like* versions of objects.

· *Main point*:

The *multiplicative monoid* and *abstract group* structures (but *not* the ring structures!) are *common* (cf. *"logical AND $\wedge$!"*) to †, ‡.

On the other hand, once one *deletes* the *labels* "†", "‡" to secure a "common R", one obtains a *meaningless* situation, where the common glued data may be related via "†" *OR "$\vee$"* via "‡" to the common R, but *not simultaneously* to both!

· When $R = \mathbb{Z}$ (or, in fact, more generally, the *ring of integers* "$\mathcal{O}_F$" of a number field F — cf. the multiplicative *norm map* $\mathrm{N}_{F/\mathbb{Q}} : F \to \mathbb{Q}$), one may consider the *"height"*

$$\log(|x|) \in \mathbb{R}$$

for $0 \neq x \in \mathbb{Z}$. Then the *N-th power map* of (i), (ii) corresponds, after passing to *heights*, to *multiplying real numbers by N*; the *multiradial algorithm* corresponds to saying that the height is *unaffected (up to a mild error term!)* by *multiplication by N*, hence that the *height is bounded*!

§3. Analogy with the projective line/Riemann sphere
(cf. [EssLgc], Example 2.4.7; [Alien], §3.1; [EssLgc], §1.5, §3.5, §3.8, §3.9, §3.10)

· Let k be a *field* (in fact, could be taken to be an arbitrary ring), R a *k-algebra*. Denote *units* of a ring by a superscript "$\times$". Write

$\mathbb{A}^1$ for the *affine line* $\mathrm{Spec}(k[T])$ over k,

$\mathbb{G}_\mathrm{m}$ for the open subscheme $\mathrm{Spec}(k[T, T^{-1}])$ of $\mathbb{A}^1$ obtained by removing the origin.

Recall that $\mathbb{A}^1$ is equipped with a well-known natural structure of *ring scheme* over k, while $\mathbb{G}_\mathrm{m}$ is equipped with a well-known natural structure of *(multiplicative) group scheme* over k. Moreover, we observe that the standard coordinate T on $\mathbb{A}^1$ and $\mathbb{G}_\mathrm{m}$ determines *natural bijections*:

$$\mathbb{A}^1(R) \xrightarrow{\sim} R, \quad \mathbb{G}_\mathrm{m}(R) \xrightarrow{\sim} R^\times$$

· Write ${}^\dagger\mathbb{A}^1$, ${}^\ddagger\mathbb{A}^1$ for the *k-ring schemes* given by *copies* of $\mathbb{A}^1$ equipped with *labels* "$\dagger$", "$\ddagger$". Observe that there exists a *unique isomorphism* of *k-ring schemes* ${}^\dagger\mathbb{A}^1 \xrightarrow{\sim} {}^\ddagger\mathbb{A}^1$; moreover, there exists a *unique isomorphism* of *k-group schemes*

$$(-)^{-1} : {}^\dagger\mathbb{G}_\mathrm{m} \xrightarrow{\sim} {}^\ddagger\mathbb{G}_\mathrm{m}$$

that maps ${}^\dagger T \mapsto {}^\ddagger T^{-1}$. Note that $(-)^{-1}$ does *not extend* to an isomorphism ${}^\dagger\mathbb{A}^1 \xrightarrow{\sim} {}^\ddagger\mathbb{A}^1$ and is clearly *not compatible* with the *k-ring scheme structures* of ${}^\dagger\mathbb{A}^1$ ($\supseteq {}^\dagger\mathbb{G}_\mathrm{m}$), ${}^\ddagger\mathbb{A}^1$ ($\supseteq {}^\ddagger\mathbb{G}_\mathrm{m}$).

· The *standard construction* of the *projective line* $\mathbb{P}^1$ may be understood as the result of *gluing* ${}^{\dagger}\mathbb{A}^1$ to ${}^{\ddagger}\mathbb{A}^1$ along the isomorphism

$$ {}^{\dagger}\mathbb{A}^1 \supseteq {}^{\dagger}\mathbb{G}_{\mathrm{m}} \stackrel{(-)^{-1}}{\longrightarrow} {}^{\ddagger}\mathbb{G}_{\mathrm{m}} \subseteq {}^{\ddagger}\mathbb{A}^1 $$

— i.e., at the level of *R-rational points*

$$ {}^{\dagger}R \supseteq {}^{\dagger}R^{\times} \stackrel{(-)^{-1}}{\longrightarrow} {}^{\ddagger}R^{\times} \subseteq {}^{\ddagger}R $$

— where ${}^{\square}R = {}^{\square}\mathbb{A}^1(R)$, ${}^{\square}R^{\times} = {}^{\square}\mathbb{G}_{\mathrm{m}}(R)$, for $\square \in \{\dagger, \ddagger\}$ (cf. the *gluing* situation discussed in §2, where "$(-)^{-1}$" corresponds to "$(-)^N$"!). Thus, *relative to this gluing*, we observe that there exists a *single rational function* on the copy of "$\mathbb{G}_{\mathrm{m}}$" that appears in the gluing that is *simultaneously* equal to the rational function ${}^{\dagger}T$ on ${}^{\dagger}\mathbb{A}^1$ *AND* [cf. "$\wedge$"!] to the rational function ${}^{\ddagger}T^{-1}$ on ${}^{\ddagger}\mathbb{A}^1$.

· *Summary*:

The standard construction of the *projective line* may be regarded as consisting of a *gluing* of two *ring schemes* along an *isomorphism* of *multiplicative group schemes* that is *not compatible* with the *ring scheme* structures on either side of the gluing.

Finally, we observe that if, in the gluing under discussion, one *arbitrarily deletes* the *distinct labels* "†", "‡" (e.g., on the grounds that both ring schemes represent "THE" structure sheaf "$\mathcal{O}_X$" of a k-scheme X!), then the resulting *"gluing without labels"* amounts to a gluing of a *single copy* of $\mathbb{A}^1$ to itself that maps the standard coordinate T on $\mathbb{A}^1$ (regarded, say, as a rational function on $\mathbb{A}^1$) to T^{-1}. That is to say, such a *deletion of labels* (even when restricted to the (abstractly isomorphic) multiplicative monoids ${}^{\dagger}T^{\mathbb{Z}}$, ${}^{\ddagger}T^{\mathbb{Z}}$!) immediately results in a *contradiction* (i.e., since $T \neq T^{-1}$!), unless one passes to some sort of *quotient* of $\mathbb{A}^1$. On the other hand, passing to such a quotient amounts, from a foundational/logical point of view, to the introduction of some sort of *indeterminacy*, i.e., to the consideration of some sort of *collection of possibilities* [cf. "$\vee$"!].

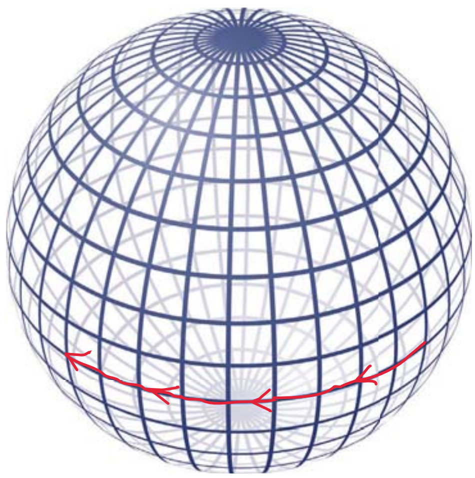
· When $k = \mathbb{C}$ (i.e., the *complex number field*), one may think of the projective line $\mathbb{P}^1$ as the *Riemann sphere* $\mathbb{S}^2$ equipped with the *Fubini-Study metric* and of the gluing under discussion as the gluing, along the *equator* $\mathbb{E}$, of the *northern hemisphere* $\mathbb{H}^+$ to the *southern hemisphere* $\mathbb{H}^-$. Then the discussion above of the standard coordinates "$^\dagger T$", "$^\ddagger T$" translates into the following (at first glance, *self-contradictory*!) phenomenon:

> an *oriented flow* along the *equator* — which may be thought of physically as a sort of *east-to-west wind current* — appears *simultaneously* to be flowing in the *clockwise* direction, from the point of view of $\mathbb{H}^+ \subseteq \mathbb{S}^2$, *AND* in the *counterclockwise* direction, from the point of view of $\mathbb{H}^- \subseteq \mathbb{S}^2$.

In particular, if one *arbitrarily deletes the labels* "+", "−" and *identifies* $\mathbb{H}^-$ with $\mathbb{H}^+$, then one does indeed literally obtain a *contradiction*. On the other hand, one may relate $\mathbb{H}^-$ to $\mathbb{H}^+$ (*not* by such an arbitrary deletion of labels (!), but rather) by applying

> the metric/geodesic geometry of $\mathbb{S}^2$ — i.e., by considering the *geodesic flow* along *great circles/lines of longitude* — to *represent*, up to a *relatively mild distortion*, the entirety of $\mathbb{S}^2$, i.e., including $\mathbb{H}^- \subseteq \mathbb{S}^2$, as a sort of *deformation/displacement* of $\mathbb{H}^+$ (cf. the point of view of *cartography*!).

It is precisely this metric/geodesic approach that corresponds to the *anabelian geom.*-based *multiradial algorithm for the* Θ*-pilot* in IUT (cf. the analogy discussed in [Alien], §3.1, (iv), (v), as well as in [EssLgc], §3.5, §3.10, between *multiradiality* and *connections/parallel transport/crystals*!).



northern hemisphere $\mathbb{H}^+$

——— *equator* $\mathbb{E}$ ———

southern hemisphere $\mathbb{H}^-$

· In this context, it is important to remember that, just like SGA, IUT is *formulated entirely in the framework* of

"ZFCG"

(i.e., ZFC + Grothendieck's axiom on the existence of universes), especially when considering various *set-theoretic/foundational* subtleties (?) of *"gluing"* operations in IUT (cf. [EssLgc], §1.5, §3.8, §3.9, as well as [EssLgc], §3.10, especially the discussion of *"log-shift adjustment"* in (Stp 7)):

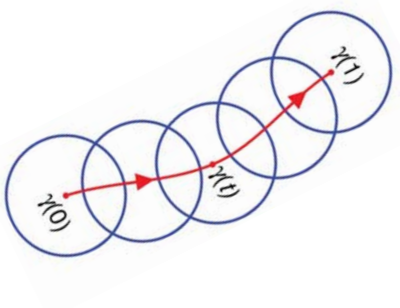
· gluing is performed at the abstract level of *diagrams* (cf. graph of groups/anabelioids), is *not* equipped with an *embedding* into some *familiar ambient space* (like a sphere);

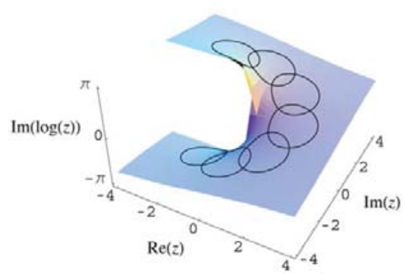
· *output of reconstruction algorithms* only well-defined at the level of *objects up to isomorphism* (+ *suitable indeterminacies*), i.e., "types/packages of data" (such as groups, rings, monoids, diagrams, etc.) called *"species"*,

$\Longrightarrow$ hence the (subtle?) importance of *closed loops* in order to obtain *set-theoretic comparisons* that are *not possible at intermediate steps*

... note importance of working with *"types/packages of data"* (cf., e.g., the *diagrams* referred to above!) — as opposed to certain particular underlying sets of interest! — cf. the classical functoriality of *resolutions* in cohomology, as well as *algebraic closures* of fields up to *conjugacy indeterminacies* (which become unnecessary, e.g., if one considers *norms*!)

... note importance of working with *"closed loops"* — cf. *norms* in Galois theory, as well as the classical theory of *analytic continuation/Riemann surfaces* (which is reminiscent of the classical *Riemann-Weierstrass* dispute!), the *geodesic completeness/closed geodesics* of the sphere.





§4. **Brief preview of the Galois-orbit version of IUT**
(cf. "Expanding Horizons" *videos/slides* cited in §1; [GSCsp]; [AnPf]; [Alien], §3.11, (iii))

· Brief preview of various *new enhanced versions of IUT*, which is closely related to recent progress (joint work in progress!) on the *Section Conjecture ("SC")*:

· [GSCsp]: reduces, using *RNS* (cf. [RNSPM]), together with a result of Stoll, geometricity of an arbitrary Galois section of a hyperbolic curve over a number field to

· *local geometricity* at each nonarchimedean prime, plus

· *3 global conditions*, which correspond, respectively, to *3 new enhanced versions of IUT*!

· [GSCsp]+[AnPf]: substantial progress on the *p-adic SC* that is closely related to the use of *Raynaud-Tamagawa "new-ordinariness"* in the theory of *RNS* (cf. [RNSPM]), which functions as a sort of *local analogue of IUT* — via the analogy

$$\text{“}N \cdot (-) \approx (-)\text{”} \longleftrightarrow \text{“Norm}(-) = (-)\text{”!}$$

· One such new enhanced version of IUT is the *Galois-orbit version of IUT (GalOrbIUT)*, which implies:

· one of the 3 global conditions mentioned above in the discussion of the *Section Conjecture* (*"intersection-finiteness"*);

· *nonexistence of Siegel zeroes* of Dirichlet L-functions associated to imaginary quadratic number fields (i.e., by applying the work of Colmez/Granville-Stark/Táfula);

· *numerically stronger* version of *abc/Szpiro* inequalities.

· That is to say, we obtain three *a priori different* applications to

· *anabelian geometry* ("local-global" Section Conjecture),
· *analytic number theory* (nonexistence of Siegel zeroes),
· *diophantine geometry* (abc/Szpiro inequalities)

— a *striking example* of *Poincaré's quote*, i.e., all three are essentially the *same mathematical phenomenon* of *bounding heights*, i.e., *bounding "local denominators"!*

- Here, the *local-global Section Conjecture* application is also noteworthy in that
  - it exhibits IUT as "*anabelian geometry* applied to obtain more *anabelian geometry!*" (less psychologically/intuitively surprising than the other two applications!);
  - it is *technically the most difficult/essential* (!) of the three, i.e., to the extent that the *other two* applications may be thought of, to a substantial extent, as being *"inessential by-products"*;
  - the *historical point of view* (cf., e.g., of Grothendieck's famous "letter to Faltings") suggests (*without any proof!*) that the Section Conjecture might imply results in diophantine geometry (such as the Mordell Conjecture).

- In this context, it is interesting to recall (cf. [Alien], §3.11, (iii)) that the essential content of *anabelian geometry* may be understood as a sort of *"conceptual translation"* of the *abc inequality*:
  - *anabelian geometry*:

    addition reconstructed from *multiplication*

    [i.e., *addition* "dominated by" *multiplication*!]
  - *abc inequality*:

    height ("additive size") $\lesssim$ *conductor ("multiplicative size")*

    [i.e., *addition* "dominated by" *multiplication*!]

  . . . cf. *conceptual Weil Conjectures* versus *numerical inequalities* for the number of rational points of a variety over a finite field!

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